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CONSTRUCTIONS OF p-ADIC L-FUNCTIONS AND ADMISSIBLE MEASURES FOR HERMITIAN MODULAR FORMS

ALEXEI PANTCHICHKINE

Dedicated to Professor Mirjana Vuković on the occasion of her 70th birthday

ABSTRACT. For a prime p and a positive integer n, the standard zeta function $L_F(s)$ is considered, attached to an Hermitian modular form $F = \sum_H A(H)q^H$ on the Hermitian upper half plane $\mathcal{H}_n$ of degree n, where H runs through semi-integral positive definite Hermitian matrices of degree n, i.e. $H \in \Lambda_n(\mathcal{O})$ over the integers $\mathcal{O}$ of an imaginary quadratic field K, where $q^H = \exp(2\pi i \mathrm{Tr}(HZ))$. Analytic p-adic continuation of their zeta functions constructed by A.Bouganis in the ordinary case (in [Bou16] is presently extended to the admissible case via growing p-adic measures. Previously this problem was solved for the Siegel modular forms, [CourPa], [BS00]. Present main result is stated in terms of the Hodge polygon $P_H(t) : [0,d] \to \mathbb{R}$ and the Newton polygon $P_N(t) = P_{N,p}(t) : [0,d] \to \mathbb{R}$ of the zeta function $L_F(s)$ of degree $d = 4n$. Main theorem gives a p-adic analytic interpolation of the L values in the form of certain integrals with respect to Mazur-type measures.

1. INTRODUCTION

1.1. p-adic zeta functions of modular forms

Since the p-adic zeta function of Kubota-Leopoldt was constructed by p-adic interpolation of zeta-values $\zeta(1-k) = -B_k/k (k \geq 1)$ [KuLe64], also p-adic zeta functions of various modular forms were constructed, such as p-adic interpolation of the special values

$$L_\Delta(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\tau(n)n^{-s},\ (s = 1, 2, \cdots, 11),\ \ \Delta = \sum_{n=1}^{\infty} \tau(n)q^n,$$

for the Ramanujan function $\tau(n)$ twisted by Dirichlet characters $\chi : (\mathbb{Z}/p^r\mathbb{Z})^* \to \mathbb{C}^*$. Interpolation done in the elliptic and Hilbert modular cases by

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Yu.I.Manin and B.Mazur, via modular symbols and p-adic integration, see [Ma73], [Ma76]).

In the Siegel modular case $Sp(2n,\mathbb{Z})$ the p-adic standard zeta functions were constructed in [Pa88], [Pa91] via Andrianov's identity of Rankin-Selberg type (n even), and [BS00] via doubling method.

1.2. Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s,\mathbf{f})$ (definitions)

Let $\theta=\theta_K$ be the quadratic character attached to $K, n'=\left[\frac{n}{2}\right]$.

$$\Gamma_{n,K}=\left\{M=\begin{pmatrix}A\ B\\ C\ D\end{pmatrix}\in GL_{2n}(\mathcal{O}_K)|M\eta_nM^*=\eta_n\right\},\ \eta_n=\begin{pmatrix}0_n & -I_n\\ I_n & 0_n\end{pmatrix},$$

$$\mathcal{Z}(s,\mathbf{f})=\left(\prod_{i=1}^{2n}L(2s-i+1,\theta^{i-1})\right)\sum_{\mathfrak{a}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a})=\lambda(\mathfrak{a})\mathbf{f},\mathfrak{a}\subset\mathcal{O}_K$)

$=\prod_{\mathfrak{q}}\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q}\subset\mathcal{O}_K$,

with $\deg\mathcal{Z}_{\mathfrak{q}}(X)=2n,$ the Satake parameters $t_{i,\mathfrak{q}}, i=1,\cdots,n$),

$\boxed{\mathcal{D}(s,\mathbf{f})=\mathcal{Z}(s-\frac{\ell}{2}+\frac{1}{2},\mathbf{f})}$ (Motivically normalized standard zeta function

with a functional equation $s\mapsto\ell-s;$ $\mathrm{r}k=4n,$ and motivic weight $\ell-1$).

Main result: p-adic interpolation of all critical values $\mathcal{D}(s,\mathbf{f},\chi)$ normalized by $\times\Gamma_{\mathcal{D}}(s)/\Omega_{\mathbf{f}}$, in the critical strip $n\le s\le\ell-n$ for all χ mod p^r in both bounded or unbounded case , i.e. when the product $\alpha_{\mathbf{f}}=\left(\prod_{\mathfrak{q}|p}\prod_{i=1}^{n}t_{\mathfrak{q},i}\right)p^{-n(n+1)}$ is not a p-adic unit.

1.3. The idea of motivic normalization: Ikeda's lifting [Ike08]

The standard Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of weight ℓ, used as a pattern, namely

$S_{2k+1}(\Gamma_0(D),\theta)\ni f\rightsquigarrow\mathbf{f}=Lift(f)\in\mathcal{S}_{2k+2n'}(\Gamma_{K,n}),$ if $n=2n'$ is even (E)

$S_{2k}(\mathrm{SL}(\mathbb{Z}))\ni f\rightsquigarrow\mathbf{f}=Lift(f)\in\mathcal{S}_{2k+2n'}(\Gamma_{K,n}),$ if $n=2n'+1$ is odd (O)

the standard $L-$ function of $\mathbf{f}=Lift^{(n)}(f)$ is $\mathcal{Z}(s,\mathbf{f})=$

$$\prod_{i=1}^{n}L(s+k+n'-i+(1/2),f)L(s+k+n'-i+(1/2),f,\theta)\ \text{[Ike08]}$$

$$=\prod_{i=0}^{n-1}L(s+\ell/2-i-(1/2),f)L(s+\ell/2-i-(1/2),f,\theta).$$

because in the lifted case $k+n' = \ell/2$, and the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1-s$ becomes $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

This Gamma factor suggests the following motivic normalization $\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s-(\ell/2)+(1/2),\mathbf{f})$ with the Gamma factor

$$\Gamma_{\mathcal{Z}}(s-(\ell/2)+(1/2)) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s-i)^2,$$

so that for Ikeda's lifting case there is a nice identity

$$\mathcal{D}(s,\mathbf{f}) = \Gamma_{\mathcal{D}}(s,\mathbf{f}) = \prod_{i=0}^{n-1} L(s-i,f)L(s-i,f,\theta).$$

The L-function $\mathcal{D}(s)$ satisfies the symmetry $s \mapsto \ell - s$ of motivic weight $\ell-1$ with the slopes $2\cdot 0, 2\cdot 1, \ldots 2\cdot(n-1)$,
$2\cdot(\ell-n), \cdots, 2\cdot(\ell-1)$, so that Deligne's critical values are at $s=n,\ldots,s=\ell-n$. Moreover the existence of p-adic L-funcions directly follows in this case from the product (even in the admissible case (non-ordinary p, e.g. for $f=\Delta$, $p=7$, $\tau(7) = -16744 = -2^3\cdot 7\cdot 13\cdot 23$ (for any K, but $n=2n'+1$ must be odd) with $\ell = 2k+2n' = 12+2n'$.

2. General zeta functions: critical values and coefficients

More general zeta functions are Euler products of degree d

$$\mathcal{D}(s,\chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s} = \prod_p \frac{1}{\mathcal{D}_p(\chi(p)p^{-s})},\ \Lambda_{\mathcal{D}}(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi),$$

where $\deg \mathcal{D}_p(X) = d$ for all but finitely many p, and $\mathcal{D}_p(0)=1$.

In many cases algebraicity of the zeta values was proven as

$$\frac{\mathcal{D}^*(s_0,\chi)}{\Omega_{\mathcal{D}}^{\pm}} \in \mathbb{Q}(\{\chi(n),a_n\}_n),\ \text{where } \mathcal{D}^*(s,\chi) \text{ is normalized by } \Gamma_{\mathcal{D}},$$

at critical points $s_0 \in \mathbb{Z}_{crit}$ as linear combinations of coefficients a_n dividing out periods $\Omega_{\mathcal{D}}^{\pm}$, where $\mathcal{D}^*(s_0,\chi) = \Lambda_{\mathcal{D}}(s_0,\chi)$ if $h^{\ell,\ell}=0$.

In p-adic analysis, the Tate field is used $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$, the completion of an algebraic closure $\overline{\mathbb{Q}}_p$, in place of $\mathbb{C}$. Let us fix embeddings $\begin{cases} i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \\ i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \end{cases}$ and try to continue analytically these zeta values to $s \in \mathbb{Z}_p$, χ mod p^r.

2.1. **Main result stated with Hodge/Newton polygons of** $\mathcal{D}(s)$

The Hodge polygon $P_H(t):[0,d]\to\mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t):[0,d]\to\mathbb{R}$ at p are piecewise linear:
The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s\mapsto w+1-s$, relating $\Lambda_{\mathcal{D}}(s,\chi)=\Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi)$ and $\Lambda_{\mathcal{D}^\rho}(w+1-s,\overline{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s)=\Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s)=\prod_{j\le\frac{w}{2}}\Gamma_{j,w-j}(s)$, where

$$\Gamma_{j,w-j}(s)=\begin{cases}\Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j<w,\\ \Gamma_{\mathbb{R}}(s-j)^{h_+^{j,j}}\Gamma_{\mathbb{R}}(s-j+1)^{h_-^{j,j}}, & \text{if } 2j=w, \text{ where}\end{cases}$$

$\Gamma_{\mathbb{R}}(s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s)=\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)=2(2\pi)^{-s}\Gamma(s),\ h^{j,j}=h_+^{j,j}+h_-^{j,j}$, $\sum_j h^{j,w-j}=d$.

The Newton polygon at p is the convex hull of points $(i, ord_p(a_i))$ $(i=0,\dots,d)$; its slopes λ are the p-adic valuations $ord_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X)\in\overline{\mathbb{Q}}[X]\subset\mathbb{C}_p[X]$: $length_\lambda=\sharp\{i \mid ord_p(\alpha_i)=\lambda\}$.

2.2. **A** p**-adic analogue of** $L_f(s)$ **(Manin-Mazur)**

It is a p-adic analytic function $L_{f,p}(s,\chi)$ of p-adic arguments $s\in\mathbb{Z}_p$, χ mod p^r which interpolates algebraic numbers

$$L_f^*(s,\chi)/\omega^\pm\in\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}_p=\hat{\overline{\mathbb{Q}}}_p \text{ (the Tate field)}$$

for $1\le s\le k-1$, $\omega^\pm$ are periods of f (for example, in the Hurwitz case $f=g_\psi=f_m$, and $\omega^\pm=\Omega^m$). The complex analytic L function of f is defined for all $s\in\mathbb{C}$; in the absolutely convergent case $\Re(s)>(k+1)/2$, $L_f^*(s,\chi)=\Gamma_{\mathbb{C}}(s)\sum_{n\ge1}\chi(n)a_nn^{-s}$, is a holomorphic function with a functional equation $s\mapsto k-s$. According to Manin and Shimura, the ratio is algebraic if the period $\omega^\pm$ is chosen according to the parity $\chi(-1)(-1)^{-s}=\pm1$.

Application: if an elliptic curve $E/\mathbb{Q}$ with complex multiplication by K has infinitly many rational points then $L(E,1)=0$ because infinitely many primes p divide $L^*_{f_K}(1,\chi)/\Omega$, see [CoWi77].

3. p-ADIC ANALYTIC INTERPOLATION OF $\mathcal{D}(s,\mathbf{f},\chi)$

The result expresses the zeta values as integrals with respect to p-adic Mazur-type measures. These measures are constructed from the Fourier coefficients of Hermitian modular forms, and from eigenvalues of Hecke operators on the unitary group.

Pre-ordinary case: $P_H(t) = P_{N,p}(t)$ *at* $t = \frac{d}{2}$ The integrality of measures is proven by T.Bouganis [Bou16], representing $\mathcal{D}^*(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi)$ as a Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce certain bounded measures $\mu_{\mathcal{D}}$ from integral representations and Petersson product, [CourPa]. For the case of p inert in K, see [Bou16].

Admissible case: $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2}) > 0$ The zeta distributions are unbounded, but their sequence produce h-admissible (growing) measures of Amice-Vélu-type, allowing to integrate any continuous characters $y \in \mathrm{H}om(\mathbb{Z}_p^*, \mathbb{C}_p^*) = \mathcal{Y}_p$. A general result is used on the existence of h-admissible (growing) measures from binomial congruences for the coefficients of Hermitian modular forms. Their p-adic Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y(x) d\mu_{\mathcal{D}}(x)$, $\mathcal{L}_{\mathcal{D}} : \mathcal{Y}_p \to \mathbb{C}_p$ give p-adic analytic interpolation of growth $\log_p^h(\cdot)$ of the L-values: the values $\mathcal{L}_{\mathcal{D}}(\chi x_p^m)$ are integrals given by $i_p\left(\dfrac{\mathcal{D}^*(m,\mathbf{f},\chi)}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$.

3.1. **A Hermitian modular form of weight ℓ with character σ**

is a holomorphic function $\mathbf{f}$ on $\mathcal{H}_n$ $(n \geq 2)$ such that $\mathbf{f}(g\langle Z\rangle) = \sigma(g)\mathbf{f}(Z)j(g,Z)^\ell$ for any $g \in \Gamma_{n,K}$. Here σ be a character of $\Gamma_K^{(n)}$, trivial on $\left\{\begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}\right\}$, and for $Z \in \mathcal{H}_n$, put $g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$, $j(g,Z) = \det(CZ+D)$.

Fourier expansions: a semi-integral Hermitian matrix is a Hermitian matrix $H \in (\sqrt{-D_K})^{-1}M_n(\mathcal{O})$ whose diagonal entries are integral.

Denote the set of semi-integral Hermitian matrices by $\Lambda_n(\mathcal{O})$, the subset of its positive definite elements is $\Lambda_n(\mathcal{O})^+$, with $\mathcal{O} = \mathcal{O}_K$.

A Hermitian modular form $\mathbf{f}$ is called a cusp form if it has a Fourier expansion of the form $\mathbf{f}(Z) = \sum_{H \in \Lambda_n(\mathcal{O})^+} A(H)q^H$. Denote the space of cusp forms of weight ℓ with character σ by $\mathcal{S}_\ell(\Gamma_{n,K},\sigma)$.

4. The standard zeta function of a Hermitian modular form

For all integral ideals $\mathfrak{a} \subset \mathcal{O}$ let $T(\mathfrak{a})$ denotes the Hecke operator associated to it as in [Shi00], page 162, using the action of double cosets $\Gamma\xi\Gamma$ with $\xi = \mathrm{d}iag(\hat{D}, D)$, $(\det(D)) = (\alpha)$, $\hat{D} = (D^*)^{-1}$, $\alpha \in \mathfrak{a}$.

Consider a non-zero Hermitian modular form $\mathbf{f} \in Mc_\ell(\Gamma)$, for a (congruence) subgroup $\Gamma \subset \Gamma_{n,K}$, and assume $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral ideals $\mathfrak{a} \subset \mathcal{O}$. Then

$$\mathcal{Z}(s,\mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s-i+1, \theta^{i-1})\right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

the sum is over all integral ideals of O_K.

This series has an Euler product representation $\mathcal{Z}(s,\mathbf{f}) = \prod_{\mathfrak{q}}(\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$, where the product is over all prime ideals of O_K, $\mathcal{Z}_{\mathfrak{q}}(X)$ *is the numerator of the series* $\sum_{r\geq 0}\lambda(\mathfrak{q}^r)X^r \in \mathbb{C}(X)$*, computed by Shimura as follows.*

4.1. **Euler factors of the standard zeta function,** [Shi00]**, p. 171**

The Euler factors $\mathcal{Z}_{\mathfrak{q}}(X)$ in the Hermitian modular case at the prime ideal $\mathfrak{q}$ of O_K are

(i) $\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n}\Big((1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)(1-N(\mathfrak{q})^{n}t_{\mathfrak{q},i}^{-1}X)\Big)^{-1}$,

if $\mathfrak{q}^{\rho} = \mathfrak{q}$ and $\mathfrak{q} \not| \mathfrak{c}$, (the inert case outside level $\mathfrak{c}$),

(ii) $\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{2n}\Big((1-N(\mathfrak{q}_1)^{2n}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}X_1)(1-N(\mathfrak{q}_2)^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X_2)\Big)^{-1}$,

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_1^{\rho} = \mathfrak{q}_2$ and $\mathfrak{q}_i \not| \mathfrak{c}$ for $i = 1,2$ (the split case outside level) ,

(iii) $\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n}\big(1-N(\mathfrak{q})^{n-1}t_{q,i}X\big)^{-1}$, if $\mathfrak{q}^{\rho} = \mathfrak{q}$ and $\mathfrak{q}|\mathfrak{c}$ (inert level divisors),

(iv) $\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{n}\Big((1-N(\mathfrak{q}_1)^{n-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}X_1)(1-N(\mathfrak{q}_2)^{n-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X_2)\Big)^{-1}$,

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_i|\mathfrak{c}$ for $i = 1,2$ (split level divisors).

where the $t_{?,i}$ above for $? = \mathfrak{q}, \mathfrak{q}_1\mathfrak{q}_2$, are the Satake parameters of the eigenform $\mathbf{f}$.

4.2. **The standard motivic-normalized zeta** $\mathcal{D}(s,\mathbf{f},\chi)$

The standard zeta function of $\mathbf{f}$ is defined by means of the p-parameters as the following Euler product:

$$\mathcal{D}(s,\mathbf{f},\chi) = \prod_{p}\prod_{i=1}^{2n}\left\{\left(1-\frac{\chi(p)\alpha_i(p)}{p^s}\right)\left(1-\frac{\chi(p)\alpha_{4n-i}(p)}{p^s}\right)\right\}^{-1},$$

where χ is an arbitrary Dirichlet character. The p–parameters $\alpha_1(p),\ldots,\alpha_{4n}(p)$ of $\mathcal{D}(s,\mathbf{f},\chi)$ for p not dividing the level C of the form $\mathbf{f}$ are related to the the $4n$ characteristic numbers

$$\alpha_1(p),\ \cdots,\ \alpha_{2n}(p),\ \alpha_{2n+1}(p),\ \cdots,\ \alpha_{4n}(p)$$

of the product of all $\mathfrak{q}$-factors $\mathcal{Z}_{\mathfrak{q}}(N\mathfrak{q}^{(\ell-1)/2)}X)^{-1}$ for all $\mathfrak{q}|p$, which is a polynomial of degree $4n$ of the variable $X = p^{-s}$ (for almost all p) with coefficients in a number field $T = T(\mathbf{f})$.

There is a relation between the two normalizations $\mathcal{Z}(s-\frac{\ell}{2}+\frac{1}{2},\mathbf{f}) = \mathcal{D}(s,\mathbf{f})$ explained in [Ha97] for general zeta functions.

Motivically, the funcion $\mathcal{D}(s,\mathbf{f})$ should correspond to $L(s,\mathrm{Res}_{K/\mathbb{Q}}(M_\mathbf{f})\otimes\chi)$ with $\mathrm{r}k_K(M_\mathbf{f}) = 2n$, and $\mathrm{Res}_{K/\mathbb{Q}}(M_\mathbf{f}) = 4n$ (Weil restriction of scalars of a motive $M_\mathbf{f}$ over K to $\mathbb{Q}$) .

5. Description of the Main results

Let $\Omega_\mathbf{f}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s-\frac{\ell}{2}+\frac{1}{2},\mathbf{f})$ the standard zeta function, and

$$\alpha_\mathbf{f} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}, \;\; h = \mathrm{o}rd_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_\mathbf{f}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Definition 5.1. *Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then:*

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.

- $\mu : \mathcal{C}^h \to M$ is called an h admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left|\int_{a+(p^v)} (x-a)^j d\mu\right|_p \leq p^{-v(h-j)}$$

for $j = 0,1,...,h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*,\mathbb{C}_p^*)$ be the space of definition of p-adic Mellin transform

Theorem 5.1 ([Am-V], [MTT]). *For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).*

Theorem 5.2 (Main Theorem). *Let $\mathbf{f}$ be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 4n+2$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n,\cdots,\ell-n$ with the properties:*

i) *for all pairs (s,χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell-n$,*

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s,\chi)\frac{\mathcal{D}^*(s,\mathbf{f},\overline{\chi})}{\Omega_\mathbf{f}}$$

(under the inclusion i_p), with elementary factors $A_p(s,\chi) = \prod_{\mathfrak{q}|p} A_\mathfrak{q}(s,\chi)$ including a finite Euler product, Satake parameters $t_{\mathfrak{q},i}$, gaussian sums, the conductor of χ; the integral is a finite sum.

ii) *if $\mathrm{o}rd_p\left((\prod_{\mathfrak{q}|p}\prod_{i=1}^n t_{\mathfrak{q},i})p^{-n(n+1)}\right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_\mathcal{D} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*,\mathbb{C}_p^*) =: \mathcal{Y}_p$.*

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D}}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, *give bounded* p*-adic analytic interpolation of the above* L*-values to on the* $\mathbb{C}_p$*-analytic group* $\mathcal{Y}_p$*; and these distributions are related by:* $\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} d\mu^*_{\mathcal{D}}$, $X = \mathbb{Z}_p^*$, *where* $s^* = \ell - n$, $s_* = n$.

iii) *in the admissible case assume that* $0 < h \le \frac{s^* - s_* + 1}{2} = \frac{\ell + 1 - 2n}{2}$, *where* $h = \mathrm{ord}_p\left(\left(\prod_{\mathfrak{q}|p} \prod_{i=1}^n t_{\mathfrak{q},i}\right) p^{-n(n+1)}\right) > 0$, *Then there exist* h*–admissible measures* $\mu_{\mathcal{D}}$ *whose integrals* $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ *are given by* $i_p\left(A_p(s,\chi)\frac{\mathcal{D}^*(s,\mathbf{f},\overline{\chi})}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$ *with* $A_p(s,\chi)$ *as in (i); their Mellin transforms* $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, *belong to the type* $o(\log x_p^h)$.

iv) *the functions* $\mathcal{L}_{\mathcal{D}}$ *are determined by* i)-iii)

5.1. **Eisenstein series and congruences**

The (Siegel-Hermite) Eisenstein series $E_{2\ell,n,K}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined in [Ike08] by $E_{2\ell,n,K}(Z) = \sum_{g \in \Gamma_{n,K,\infty} \backslash \Gamma_{n,K}} (\det g)^\ell j(g,Z)^{-2\ell}$ (converges for $\ell > n$). The normalized Eisenstein series is given by
$\mathcal{E}_{2\ell,n,K}(Z) = 2^{-n} \prod_{i=1}^n L(i-2\ell, \theta^{i-1}) \cdot E_{2\ell,n,K}(Z)$.
If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is polynomial over $\mathbb{Z}$ in variables $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H,X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a *key point in proving congruences* for the modular forms in a *Rankin-Selberg integral*. Also, for a certain congruence subgroup $C = \Gamma_{\mathfrak{c}}$, $s \in \mathbb{C}$ and a Hecke ideal character ψ mod $\mathfrak{c}$, the series is defined

$$E(Z,s,\ell,\psi) = \sum_{g \in C_\infty \backslash C} \psi(g)(\det g)^\ell j(g,Z)^{-2\ell} |(\det g) j(g,Z)|^{-s}.$$

5.2. **An integral representation of Rankin-Selberg type**

The integral representation of Rankin-Selberg type in the Hermitian modular case: is stated for the level $\mathfrak{c}$ moodular forms:

Theorem 5.3 ((Shimura, Klosin, see [Bou16], p.13.)). *Let* $0 \neq \mathbf{f} \in Mc_\ell(\Gamma_{\mathfrak{c}}, \psi))$ *of scalar weight* ℓ, ψ mod $\mathfrak{c}$, *such that* $\forall \mathfrak{a}, \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$, *and assume that* $2\ell \ge n$, *then there exists* $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ *and* $\mathcal{R} \in \mathrm{GL}_n(K)$ *such that*

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s+3n/2,\mathbf{f},\chi) =$$
$$\Lambda_{\mathfrak{c}}(s+3n/2,\theta\psi\chi)\cdot C_0\langle \mathbf{f},\theta_{\mathcal{T}}(\chi)\mathcal{E}(\bar{s}+n,\ell-\ell_\theta,\chi^\rho\psi)\rangle_{C\sec},$$

where $\mathcal{E}(Z,s,\ell-\ell_\theta,\psi)_{C\sec}$ is a normalized group theoretic (or adelic) Eisenstein series with components as above of level $\mathfrak{c}\sec$ divisible by $\mathfrak{c}$, and weight $\ell-\ell_\theta$. Here $\langle\cdot,\cdot\rangle_{C\sec}$ is the normalized Petersson inner product associated to the congruence subgroup $C\sec$ of level $\mathfrak{c}\sec$.

$$\Gamma((s)) = (4\pi)^{-n(s+h)}\Gamma_n^\iota(s+h), \Gamma_n^\iota(s) = \pi^{\frac{n(n-1)}{2}}\prod_{j=0}^{n-1}\Gamma(s-j),$$

where $h=0$ or 1, C_0 the index of a subgroup. □

From this integral one obtains the Gamma factors for $\mathcal{D}(s)$: $\Gamma_{\mathcal{D}}(s) = L_\infty(s,\mathbf{f},\chi) = = \prod_{j=0}^{n-1}\Gamma_{\mathbb{C}}(s-j)^2$, with the symmetry $s\mapsto \ell-s$.

These factors suggest the following form of the Hodge polygon of $\mathcal{D}(s,\mathbf{f},\chi)$ of rank $d=4n$ as that of the Hodge numbers $h^{j,w-j}$ below (in the increasing order of slopes j, of weight $w=\ell-1$):

$$2\cdot(0,\ell-1),\ldots,2\cdot(n-1,\ell-n),$$
$$2\cdot(\ell-n,n-1),\ldots,2\cdot(\ell-1,0),$$

following Serre's recipe [Se70], p.11.

5.3. **Proof of the Main Theorem (ii): Kummer congruences**

Let us se the notation $\mathcal{D}_p^{alg}(m,\mathbf{f},\chi) = A_p(s,\chi)\dfrac{\mathcal{D}^*(m,\mathbf{f},\chi)}{\Omega_{\mathbf{f}}}$

The integrality of measures is proven representing $\mathcal{D}_p^{alg}(m,\chi)$ as Rankin-Selberg type integral at critical points $s=m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce bounded measures $\mu_{\mathcal{D}}$ whose construction reduces to congruences of Kummer type between the Fourier coefficients of modular forms, see also [Bou16]. Suppose that we are given infinitely many "critical pairs" (s_j,χ_j) at which one has an integral representation $\mathcal{D}_p^{alg}(s_j,\mathbf{f},\chi_j) = A_p(s,\chi)\dfrac{\langle \mathbf{f},h_j\rangle}{\Omega_{\mathbf{f}}}$ with all $h_j=\sum_{\mathcal{T}} b_{j,\mathcal{T}}q^{\mathcal{T}}\in Mc$ in a certain finite-dimensional space Mc containing $\mathbf{f}$ and defined over $\overline{\mathbb{Q}}$. We prove the following Kummer-type congruences:

$$\forall x\in\mathbb{Z}_p^*,\ \sum_j \beta_j\chi_j x^{k_j}\equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j\mathcal{D}_p^{alg}(s_j,\mathbf{f},\chi)\equiv 0 \mod p^N$$
$$\beta_j\in\overline{\mathbb{Q}}, k_j = s^*-s_j,\ \text{where } s^*=\ell-n \text{ in our case.}$$

5.4. **Admissible Hermitian case**

Let $\mathbf{f} \in \mathcal{S}_\ell(C,\psi)$ be a Hecke eigenform for the congruence subgroup $C = \Gamma_c$ of level $\mathfrak{c}$. Let $\mathfrak{q}$ be a prime of K over p, which is inert over $\mathbb{Q}$. Then we say that $\mathbf{f}$ is pre-ordinary at $\mathfrak{q}$ if there exists an eigenform $0 \neq \mathbf{f}_0 \in Mc_{\{p\}} \subset \mathcal{S}_\ell(Cp,\psi)$ with Satake parameters $t_{\mathfrak{q},i}$ such that

$$\left\| \left(\prod_{i=1}^{n} t_{\mathfrak{q},i} \right) N(\mathfrak{q})^{-\frac{n(n+1)}{2}} \right\|_p = 1,$$

where $\| \|_p$ the normalized absolute value at p.

The admissible case corresponds to

$$\left\| \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right\|_p = p^{-h} \text{ for a positive } h > 0.$$

An interpretation of h as the difference $h = P_{N,p}(d/2) - P_H(d/2)$ comes from the above explicit relations.

5.5. **Existence of h-admissible measures**

of Amice-Vélu-type gives an unbounded p-adic analytic interpolation of the L-values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^n t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in $\mathcal{O}$:

$$\mathrm{ord}_p \left(\prod_{\mathfrak{q}|p} \left(\prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right) = h.$$

We use an easy condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in the semigroup algebra $\mathcal{O}[[q]] = \mathcal{O}[[q^H]]_{H \in \Lambda(\mathcal{O})^+}$ as in Theorem 4.8 of [CourPa]. It suffices to check congruences of the type (with $\varkappa = 4$)

$$U^{\varkappa v} \Big(\sum_{j'=0}^{j} \binom{j}{j'} (-a_p^0)^{j-j'} \Phi_{j'}(a + (p^v)) \Big) \in C p^{vj} \mathcal{O}[[q]]$$

for all $j = 0, 1, \ldots, \varkappa h - 1$. Here $s = s^* - j'$, $\Phi_{j'}(a + (p^v))$ a certain convolution of two Hermitian modular forms, i.e.

$$\Phi_{j'}(\chi) = \theta(\chi) \cdot \mathcal{E}(s,\chi)$$

of a Hermitian theta series $\theta(\chi)$ and an Eisenstein series $\mathcal{E}(s,\chi)$ with any Dirichlet character χ mod p^r. We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p^*$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa].

5.6. **Proof of the Main Theorem (iii): (admissible case)**

Using a Rankin-Selberg integral representation for $\mathcal{D}^{alg}(s,\mathbf{f},\chi)$ and an eigenfunction $\mathbf{f}_0$ of Atkin's operator $U(p)$ of eigenvalue $\alpha_{\mathbf{f}}$ on $\mathbf{f}_0$ the Rankin-Selberg integral of $\mathcal{F}_{s,\chi} := \theta(\chi)\cdot\mathcal{E}(s,\chi)$ gives

$$\mathcal{D}^{alg}(s,\mathbf{f},\chi) = \frac{\langle \mathbf{f}_0, \theta(\chi)\cdot\mathcal{E}(s,\chi)\rangle}{\langle \mathbf{f},\mathbf{f}\rangle} \text{ (the Petersson product on } G = GU(\eta_n))$$
$$= \alpha_{\mathbf{f}}^{-v}\frac{\langle \mathbf{f}_0, U(p^v)(\theta(\chi)\cdot\mathcal{E}(s,\chi))\rangle}{\langle \mathbf{f},\mathbf{f}\rangle} = \alpha_{\mathbf{f}}^{-v}\frac{\langle f_0, U(p^v)(\mathcal{F}_{s,\chi})\rangle}{\langle \mathbf{f},\mathbf{f}\rangle}.$$

Modification in the admissible case: instead of Kummer congruences, to estimate p-adically the integrals of test functions: $M = p^v$: $\int_{a+(M)} (x-a)^j d\mathcal{D}^{alg} := \sum_{j'=0}^{j} \binom{j}{j'} (-a)^{j-j'} \int_{a+(M)} x^{j'} d\mathcal{D}^{alg}$, using the orthogonality of characters and the sequence of zeta distributions $\int_{a+(M)} x^j d\mathcal{D}^{alg} = \frac{1}{\sharp(O/MO)^\times}\sum_{\chi \bmod M}\chi^{-1}(a)\cdot$ $\int_X \chi(x)x^j d\mathcal{D}^{alg}$, $\int_X \chi d\mathcal{D}^{alg}_{s^*-j} = \mathcal{D}^{alg}(s^*-j,f,\chi) =: \int_X \chi(x)x^j d\mathcal{D}^{alg}$.

5.7. **Congruences between the coefficients of the Hermitian modular forms**

In order to integrate any locally-analytic function on X, it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form $\mathcal{F}_{s^*-j,\chi} = \sum_\xi v(\xi, s^*-j,\chi)q^\xi$: for $v \gg 0$, and a constant C

$$\frac{1}{\sharp(O/MO)^\times}\sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\sum_{\chi \bmod M}\chi^{-1}(a)v(p^v\xi, s^*-j',\chi)q^\xi$$
$$\in Cp^{vj}O[[q]] \quad \text{(This is a quasimodular form if } j' \neq s^*)$$

The resulting measure $\mu_{\mathcal{D}}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \mathrm{Hom}_{cont}(X,\mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.

Its p-adic Mellin transform $\mathcal{L}_{\mu_{\mathcal{D}}}$ is an analytic function on $\mathcal{Y}_p$ of the logarithmic growth $O(\log^h)$, $h = \mathrm{ord}_p(\alpha)$. □

5.8. **Appendix A. Rewriting the local factor at p with character θ**

Notice that if θ is the quadratic character attached to $K/\mathbb{Q}$ then

$$(1-\alpha_p X)(1-\alpha_p\theta(p)X) = \begin{cases} (1-\alpha_p X)^2 & \text{if } \theta(p)=1, p\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2, N(\mathfrak{q}_i)=p,\\ (1-\alpha_p^2X^2), & \text{if } \theta(p)=-1, p\mathfrak{r}=\mathfrak{q}, N(\mathfrak{q})=p^2,\\ (1-\alpha_p X) & \text{if } \theta(p)=0, p\mathfrak{r}=\mathfrak{q}^2, N(\mathfrak{q})=p.\end{cases}$$

Thus, if $X = p^{-s}$, $X^2 = p^{-2s}$, $N(\mathfrak{q}) = p$, $\mathcal{Z}_{\mathfrak{q}}(X)^{-1}$

$$= \begin{cases} \prod_{i=1}^{2n}(1 - N(\mathfrak{q}_1)^{2n} t^{-1}_{\mathfrak{q}_1\mathfrak{q}_2,i} X)(1 - N(\mathfrak{q}_2)^{-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X), & \text{if } \theta(p) = 1, \\ \prod_{i=1}^{n}(1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X^2)(1 - N(\mathfrak{q})^n t^{-1}_{\mathfrak{q},i} X^2), & \text{if } \theta(p) = -1, \\ \prod_{i=1}^{n}(1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X)(1 - N(\mathfrak{q})^n t^{-1}_{\mathfrak{q},i} X), & \text{if } \theta(p) = 0. \end{cases}$$

$$= \begin{cases} \prod_{i=1}^{n}(1 - \gamma_{p,i} X)^2 \prod_{i=1}^{n}(1 - \delta_{p,i} X)^2 & \text{if } \theta(p) = 1, \text{ i.e. } p\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2, \\ \prod_{i=1}^{n}(1 - \alpha^2_{p,i} X^2) \prod_{i=1}^{n}(1 - \beta^2_{p,i} X^2), & \text{if } \theta(p) = -1, \text{ i.e. } p\mathfrak{r} = \mathfrak{q}, \\ \prod_{i=1}^{n}(1 - \alpha'_{p,i} X) \prod_{i=1}^{n}(1 - \beta'_{p,i} X) & \text{if } \theta(p) = 0, \text{ i.e. } p\mathfrak{r} = \mathfrak{q}^2, \end{cases}$$

where $\alpha'_{p,i} = p^{n-1} t_{\mathfrak{q},i}$, $\beta'_{p,i} p^n t^{-1}_{\mathfrak{q},i}$, $\gamma_{p,i} = p^{2n} t^{-1}_{\mathfrak{q}_1\mathfrak{q}_2,i}$, $p^{-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i}$. It follows that $\prod_{\mathfrak{q}|p} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-n-(1/2)} X) = X^{4n} + \cdots$

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Alexei Pantchichkine
Institut Fourier
University Grenoble-Alpes
Gières, France
e-mail: *Alexei.Pantchichkine@ujf-grenoble.fr*